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Efficient simulation of optical wireless channel

Application to WBANs with MISO link

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Abstract

This paper presents a new optimized simulation algorithm of the optical wireless channel. It has a faster convergence for SISO (Single Input Single Output) systems, while it significantly reduces computation time for MISO (Multiple Inputs Single Output) systems, which are the main ones in WBANs (Wireless Body Area Networks). Our algorithm is based on Monte Carlo methods, and uses 3D launched rays. Its main difference with previous solutions relies on reversing the process of propagation of optical waves, to solve efficiently and in real environments the global illumination equation modeling the propagation of light. Experimental results and comparisons with the literature are given in terms of precision and computation time.

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1. Introduction

Optical wireless communications have begun in the end of XVIIIth century with the advent of optical telegraph. Today, the global market for wireless transmission is based primarily on radio telecommunication technologies. However, they cannot absorb alone the growing customer demands, firstly, for higher data rate and secondly for higher security, because they have a limited spectral bandwidth and a sensitivity to interferences. For these reasons, OWCL (Optical Wireless Channel Link) was proposed as an alternative solution to replace the radio links in several application contexts [1], especially in the health domain and hospital environments [2], [3], for three main reasons: i) the optical frequency spectrum is free, ii) the OWCL is naturally secure because the optical waves do not penetrate walls and they cannot easily be intercepted, and finally iii) the optical waves are harmless for the human being. OWCL requires an accurate characterization of the channel in order to qualify the performance limits and to design the best strategies for wireless optical links (i.e., to determine the best locations and orientations of sensors). This characterization is based on the knowledge of the impulse response of the optical channel by performing experimental measurements that can be long and costly, or by simulating the behavior of the propagation channel in realistic environments.

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All optical wireless channel simulators seek to solve the equation of global illumination or rendering equation (i.e., the equivalent of Maxwell's equations). This equation is used in computer graphics to solve the problem of light pattern through a simulation scene. It describes interactions between the light emitted by a source and the elements constituting the scene. Among others methods, we are particularly interested in Monte Carlo ones. Associated to the ray-tracing algorithm [4], they can be used to solve the rendering equation.

In this article, we first express the light propagation using the potential equations [5], [6]. Then, while we retrieve a well-known previous algorithm [7], our formalism allows to add easily some reduction variance techniques, as the Next Event Estimation and the importance sampling. This first solution, called Ray Shooting algorithm, is based on the path tracing from the transmitter. However, it is not efficient, in terms of computation time to treat MISO or mobile SISO links. Indeed it requires re-launching the same number of new rays from each transmitter in MISO case, and from each new location of the transmitter in the mobile SISO context. The MISO case is the main one in WBAN [2], [3]. One solution to this problem, being the main novelty of this paper, is to reverse the propagation direction of luminous flux: starting now by the receiver, this algorithm connects the different rays to the transmitters in a single computing step, by launching a set of common rays according to all transmitters. Thus, this algorithm is called Ray Gathering. As for the Ray Shooting algorithm, we use importance sampling and Next Event Estimation to reduce the variance, and a rejection sampling technique at the receiver, to concentrate the ray launching in its FOV (Field Of View).

This paper is organized as follow: Section II describes the general principle of Monte Carlo method and formalizes the Ray Shooting algorithm. Section III presents our new simulation algorithm, called Ray Gathering. Finally we present some results of these simulation algorithms in section IV, and compare them with previously published ones in terms of accuracy and computation times.

2. Direct Monte Carlo solution

2.1. The rendering equation

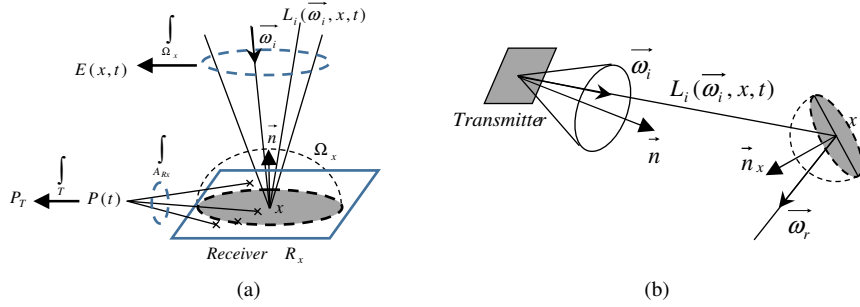


Fig. 1. (a) Photometric quantities (b) Reflection of a light flux

The modeling of light propagation requires the introduction of a number of photometric quantities. The illumination $E(x, t)$ (See Fig. 1 (a) and (1)) at point x and time t corresponds to the received power per unit area and is expressed in W/m^2 . Incident luminance in the direction $\vec{\omega}_i$ at point x and time t (See (2)) represents the received power per unit area and solid angle, and is expressed in $W/m^2/sr$. The total power $P(t)$ received at the receiver at time t , and the total received power P_T over a time interval T are finally expressed from previous quantities according to (3) and (4).

$$E(x, t) = \int_{\Omega_x} L_i(\vec{\omega}_i, x, t) |\vec{\omega}_i \cdot \vec{n}| d\omega_i, \quad (1)$$

$$L_i(\vec{\omega}_i, x, t) = \frac{dE(x, t)}{|\vec{\omega}_i \cdot \vec{n}| d\omega_i}, \quad (2)$$

$$P(t) = \int_{A_{R_x}} E(x, t) dA_x, \quad (3)$$

$$P_T = \int_T P(t) dt. \quad (4)$$

Where Ω_x is the upper hemisphere covering the point x and A_{R_x} the area of the receiver. The global illumination equation (5) governs the propagation of light waves in the environment by successive reflections:

$$L_r(\vec{\omega}_r, x, t) = L_e(\vec{\omega}_r, x, t) + \int_{\Omega_x} f_r(x, \vec{\omega}_i \rightarrow \vec{\omega}_r) L_i(\vec{\omega}_i, x, t) |\vec{\omega}_i \cdot \vec{n}| d\omega_i, \quad (5)$$

where f_r describes the reflective properties of the surface (its BRDF for Bidirectional Reflectance Distribution Function), where L_e denotes the luminance auto-emitted by this surface, L_r denotes the reflected luminance in the direction $\vec{\omega}_r$ and $\vec{\omega}_i \rightarrow \vec{\omega}_r$ the sense of rebound at the point x .

2.2. Monte Carlo methods

The stochastic Monte Carlo integration approximates the value of an integral I using a random sampling of the variable x on its integration domain [8]. Let I be any integral:

$$I = \int_{\Omega_x} f(x) dx. \quad (6)$$

In its basic form, the estimate \hat{I} of the integral I is obtained by averaging N independent samples $X_0, X_1 \dots X_N$ pondered by their choice probabilities, i.e., by their probability density function (pdf) denoted $p(x)$:

$$\hat{I} = \frac{1}{N} \sum_{i=1}^N \frac{f(X_i)}{p(X_i)}. \quad (7)$$

It is quite easy to show that the Monte Carlo estimator \hat{I} gives a correct result on average by calculating the expected value $E[\hat{I}] = I$. In other words, \hat{I} converges to the desired result of the integral. For sampling random variables we use two methods called the inversion method and the sampling rejection [8].

2.3. Ray Shooting algorithm

The ray tracing technique is a particularly promising technique [7], [4]. It is based on a precise knowledge of the simulation environment and on physical principles of the propagation theory of wireless optical waves. The idea is to launch a number of rays from the emitter in chosen directions (See Fig. 1 (b)). The propagation path of each ray is calculated involving reflections on the surfaces of the scene. At each reflection point, a new direction is randomly chosen according to the physical properties of the surface (i.e., its BRDF). Then a new reflection point is determined from the first (closest) intersection with an object in the scene (See Fig. 2 (a)). While this previous technique is quite efficient, its main drawback is that Monte Carlo integration is rather hidden. Then, classical reduction variance techniques cannot be easily plugged, while they should be helpful to accelerate its convergence. We propose to develop a formalism to enhance the link with Monte Carlo classical techniques. For this purpose, we investigate this problem using the potential function W [8], [9].

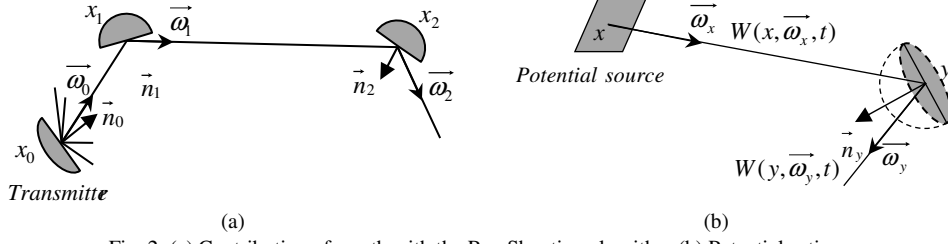


Fig. 2. (a) Contribution of a path with the Ray Shooting algorithm (b) Potential action

Let us recall that the rendering equation (5) is a recursive equation that expresses the reflected luminance at a point x in the direction $\vec{\omega}_r$ as the luminance auto emitted from x added to the sum of luminance received in all directions then reflected on x . The potential equation is also a recursive equation but describes the reverse, i.e., the action of a unit directional light source located at x , radiating in direction $\vec{\omega}_x$ [5]:

$$W(x, \vec{\omega}_x, t) = W_e(x, \vec{\omega}_x, t) + \int_{\Omega_y} f_r(y, \vec{\omega}_x \rightarrow \vec{\omega}_y) W(y, \vec{\omega}_y, t) |\vec{\omega}_y \cdot \vec{n}_y| d\vec{\omega}_y, \quad (8)$$

where W_e models a direct reception of the action of the potential source at a receiver, and y corresponds to the point seen at x from the direction $\vec{\omega}_x$ (See Fig. 2 (b)). From (1) and (8) we can calculate the expression of the illumination received at time t .

$$E(t) = \int_{A_{R_x}} \int_{\Omega_x} L_0(x, \vec{\omega}_x, t) W(x, \vec{\omega}_x, t) |\vec{\omega}_x \cdot \vec{n}_x| d\vec{\omega}_x dA_x, \quad (9)$$

where L_0 is the luminance emitted from the transmitter on x in the direction $\vec{\omega}_x$. So applying the Monte Carlo method to (9) and sampling the new directions $\vec{\omega}_j$ at each reflection according to the pdf $p_j(x_j, \vec{\omega}_j)$, we formalize the Ray Shooting algorithm by the expression of the following Monte Carlo estimator:

$$\hat{E}(t) = \frac{1}{N} \sum_{k=1}^N E_k(t), \quad E_k(t) = \frac{L_0(x_0, \vec{\omega}_0, t) |\vec{\omega}_0 \cdot \vec{n}_0|}{p_0(x_0, \vec{\omega}_0)} \cdot A_{R_x} \cdot W(x_l, \vec{\omega}_l, t) \cdot \prod_{j=1}^l \frac{f_r(x_j, \vec{\omega}_{j-1} \rightarrow \vec{\omega}_j) |\vec{\omega}_{j-1} \cdot \vec{n}_j|}{p_j(x_j, \vec{\omega}_j)}, \quad (10)$$

$$W(x_l, \vec{\omega}_l, t) = f_r(x_l, \vec{\omega}_{l-1} \rightarrow \vec{\omega}_{R_x}) V(x_l, R_x) \frac{|\vec{\omega}_{l-1} \cdot \vec{n}_l| |\vec{\omega}_{R_x} \cdot \vec{n}_{R_x}|}{\|\vec{x}_l R_x\|^2}. \quad (11)$$

where $E_k(t)$ is the contribution of a single light path, $\hat{E}(t)$ is the estimation of the total illumination received at the time t , l is the number of reflexions, $V(x_l, R_x)$ is the visibility function and A_{R_x} is the area of the receiver.

2.4. Variance reduction methods

This basic Ray Shooting algorithm has a high variance, and so a slow convergence of the results. Indeed, the probability that the last chosen direction would reach the receiver is very weak, leading to almost zero contribution. To reduce this variance, we use two variance reduction techniques. The first one is called Next Event Estimation, which principle is to attempt to connect rays to the receiver at each point of reflection, before sampling a new direction for the reflected rays (See Fig. 3). Then, the new expression of the Monte Carlo estimator, associated with Ray Shooting algorithm and the Next Event Estimation technique, becomes:

$$E_k(t) = \frac{L_0(x_0, \vec{\omega}_0, t) |\vec{\omega}_0 \cdot \vec{n}_0|}{p_0(x_0, \vec{\omega}_0)} \cdot A_{R_x} \cdot \sum_{i=1}^l \left(W(x_i, \vec{\omega}_i, t) \cdot \prod_{j=1}^{i-1} \frac{f_r(x_j, \vec{\omega}_{j-1} \rightarrow \vec{\omega}_j) |\vec{\omega}_{j-1} \cdot \vec{n}_j|}{p_j(x_j, \vec{\omega}_j)} \right) \quad (12)$$

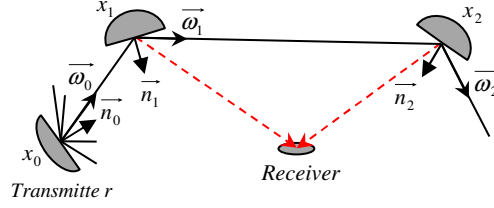


Fig. 3. Ray Shooting algorithm with Next Event Estimation

This technique produces more contributions, and so reduces the variance of the estimation. The second variance reduction method we use is importance sampling. With our formalism, it consists in using a part of integrand as a pdf. In our case, we use the cosine $|\vec{\omega}_{j-1} \cdot \vec{n}_j|$, while it can also include a part of the BRDF.

While this first algorithm is rather efficient with SIMO links, it is not the case for MISO ones. Indeed, the Ray Shooting process must be then restarted for each transmitter location, which is very costly in computation time. The same problem exists for a SISO link with a mobile transmitter. Let us notice that these two cases are the main ones in our application, where the patient is mobile in a room or lengthening on a bed with several sensors on the body, measuring vital signs. For this reason, we propose in the next section a promising solution to such MISO links, called Ray Gathering algorithm.

3. Ray Gathering algorithm

3.1. Basic algorithm

The principle of this new algorithm is to generate a path from the receiver, performing successive reflections. At each reflection a new direction is randomly chosen to recursively simulate an additional reflection. The reflected rays are tested against all the transmitters to potentially add its contribution. Thus, all the impulse responses according to each transmitter can be determined in a single computing process (See Fig. 4 (a)).

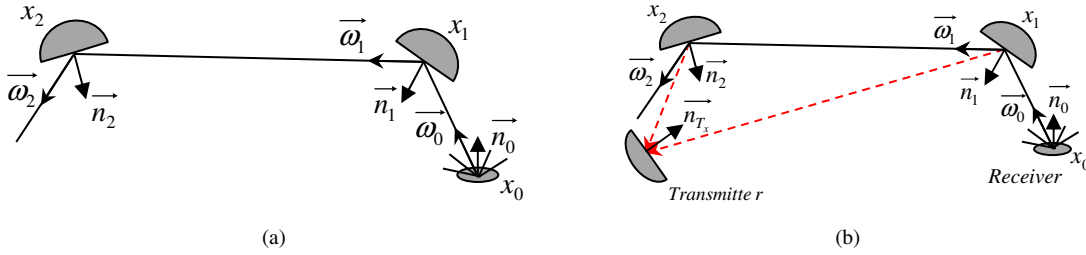


Fig. 4. (a) Contribution of a path with Ray Gathering algorithm (b) Ray Gathering algorithm with Next Event Estimation

Let us recall that the problem of global illumination is based on the rendering equation. The derivation of the basic Ray Gathering algorithm starts from the expression of the light flux in terms of luminance (1) and the rendering equation (5). The result of the estimation of the total illumination received at the time t , by the Monte Carlo method, is given by the following expression:

$$E_k(t) = \frac{|\vec{\omega}_0 \cdot \vec{n}_0|}{p_0(x_0, \vec{\omega}_0)} \cdot A_{R_x} \cdot L_i(x_l, \vec{\omega}_l, t) \cdot \prod_{j=1}^l \frac{f_r(x_j, \vec{\omega}_j \rightarrow \vec{\omega}_{j-1}) |\vec{\omega}_j \cdot \vec{n}_j|}{p_j(x_j, \vec{\omega}_j)}, \quad (13)$$

where $L_i(x_l, \vec{\omega}_l, t)$ is the luminance radiated at the last reflexion point x_l connected to the transmitter T_x :

$$L_i(x_l, \vec{\omega}_l, t) = L_0(T_x, \vec{\omega}_{T_x}, t) \cdot V(x_l, T_x) \cdot \frac{|\vec{\omega}_{l-1} \cdot \vec{n}_l| \cdot |\vec{\omega}_{T_x} \cdot \vec{n}_{T_x}|}{\|\vec{x}_l T_x\|^2}. \quad (14)$$

3.2. Variance reduction techniques

Similarly to the Ray Shooting algorithm, the Ray Gathering one presents high variance because of the weak probability to choose a random direction reaching the transmitter. So to reduce the variance we use the same techniques as with Ray Shooting: importance sampling, and Next Event Estimation technique. For the latter, we try to connect each reflection point to the different transmitters before randomly choose a new direction of reflection (See Fig. 4 (b)). Thus, the new Monte Carlo estimator of Ray Gathering algorithm is given by:

$$E_k(t) = \frac{|\vec{\omega}_0 \cdot \vec{n}_0|}{p_0(x_0, \vec{\omega}_0)} \cdot A_{R_x} \cdot \sum_{i=1}^l \left(L_i(x_i, \vec{\omega}_i, t) \cdot \prod_{j=1}^{i-1} \frac{f_r(x_j, \vec{\omega}_j \rightarrow \vec{\omega}_{j-1}) |\vec{\omega}_j \cdot \vec{n}_j|}{p_j(x_j, \vec{\omega}_j)} \right) \quad (15)$$

3.3. Rejection sampling

The Ray Gathering algorithm is based on the idea of randomly drawn directions from the receiver; this means that all the selected directions outside the FOV of the receiver carry a null luminance, and therefore do not contribute. These samples increase the variance of the estimation while they decrease the convergence speed of the algorithm, especially for low FOV (i.e., directive receiver). To reduce again the variance, we use a rejection sampling at the receiver consisting to reject directions outside the FOV, and so to only compute potentially contributory samples.

4. Results

This section presents our results and compares them to simulations presented in a reference article based on the Radiosity method [10]. This article has been a reference for many other works in the field of wireless optical systems. Its scenes are well defined, and it compares simulation results with experimental ones. We chose the configuration A for the comparison of results. The dimensions of the scene are $5 \times 5 \times 3$ m with a transmitter located at the center of the ceiling and pointing down. The receiver is placed on the floor in a corner of the room and is pointing up (See Fig. 5 (b)). The source model is Lambertian [10]. All surfaces are perfectly diffuse (Lambertian), having a reflection coefficient of 0.8 (i.e., they are few absorbent), except the floor that has a reflection coefficient of 0.3 (absorbent enough). We use for each simulation 10^8 launched rays.

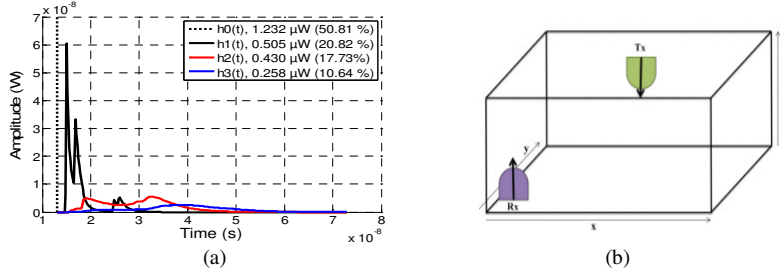


Fig. 5. (a) Results of the Ray Gathering algorithm for 0, 1, 2 and 3 reflections (b) The simulation scene

In order to compare the Ray Gathering algorithm with the Radiosity method [10] and the Ray Shooting ones, we calculate the contributions of the impulse response of order k $h^{(k)}(t)$ with k the number of reflections, which is equal to the received power over time. For these computations we have considered a transmission power equal to 1W. The comparison shows that the arrival times of rays are identical, as well as the general form of the impulse responses (See Fig. 5, 6). The total received power for each method is 2.4 mW and the percentages of received power for different orders of reflections are almost the same (See Table I). The existing difference between ours algorithms and the Radiosity method is due to the fact that the latter depends strongly on the size of the Radiosity patch, used to discretize the simulation scene. Indeed, the reduction of the size of the patches (i.e., increase the resolution of the discretization) for more precision, also drives to an exponential increase of the computation time. This is due to the fact that the complexity to compute $h^{(k)}(t)$ with the

Radiosity method is roughly proportional to N^k , N being the number of patches. This represents the major handicap of the Radiosity method. Thereafter, we present a comparative study of the three methods in terms of computing time (i.e., real computing time for Ray Shooting and Ray Gathering Algorithms). For the Radiosity results, the computation time is approximately estimated by $4 \mu\text{s} \times N^k$ (i.e., $4 \mu\text{s}$ for each elementary calculation) with N equal to 11×10^5 for computing $h^{(1)}(t)$, 44×10^3 for $h^{(2)}(t)$ and 27.5×10^2 for $h^{(3)}(t)$. Using again 27.5×10^2 patches to calculate $h^{(4)}(t)$ and $h^{(5)}(t)$ [10], although it is very small, exponentially increases computation time (See Table I).

Table 1. Comparison of computation time

Number of reflections	Radiosity method (s)	Ray Shooting Algorithm (s)	Ray Gathering Algorithm (s)	Received power Ray Gathering (μW)	Received power Radiosity (μW) [7]
$k = 0$	-	-	-	1.232 μW (50.81 %)	1.23 μW (50.6 %)
$k = 1$	4 s	52 s	51 s	0.505 μW (20.82 %)	0.505 μW (20.7 %)
$k = 2$	7.2×10^3 s	97 s	98 s	0.430 μW (17.73 %)	0.430 μW (17.7 %)
$k = 3$	$\sim 8.28 \times 10^4$ s	150 s	149 s	0.258 μW (10.64 %)	0.269 μW (11.0 %)
$k = 4$	$\sim 2.21 \times 10^8$ s	196 s	198 s	-	-
$k = 5$	$\sim 6 \times 10^{10}$ s	336 s	232 s	-	-

Table 1 shows that because of the computational complexity, the Radiosity method is limited by three reflections. On the contrary, our methods can use a greater number of reflections (See Fig. 6).

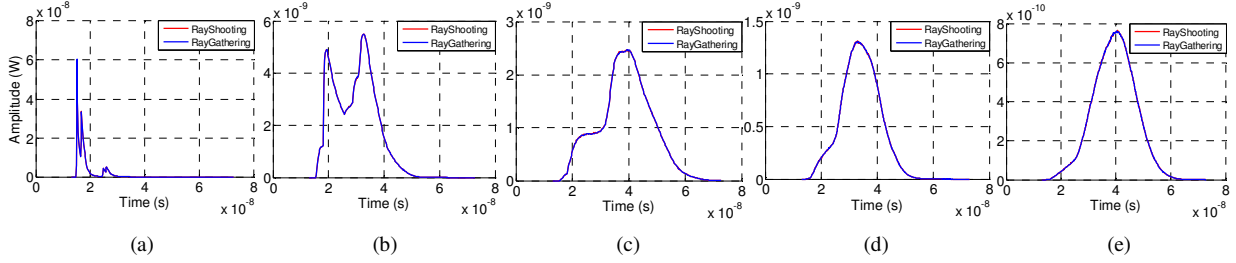


Fig. 6. Impulse response with (a) 1, (b) 2, (c) 3, (d) 4 and (e) 5 reflections

To validate the improved performances of our new Ray Gathering algorithm, we compare it with the Ray Shooting one. The first criterion of comparison is the speed of convergence that we approximate by the variance of the Monte Carlo estimations with both algorithms. Thanks to the rejection sampling, conducted for the choice of departure directions from the receiver, and concentrating all these chosen directions in the FOV of the receiver, the variance is reduced and the speed of convergence of the Ray Gathering algorithm is increased (See Fig. 7(a)).

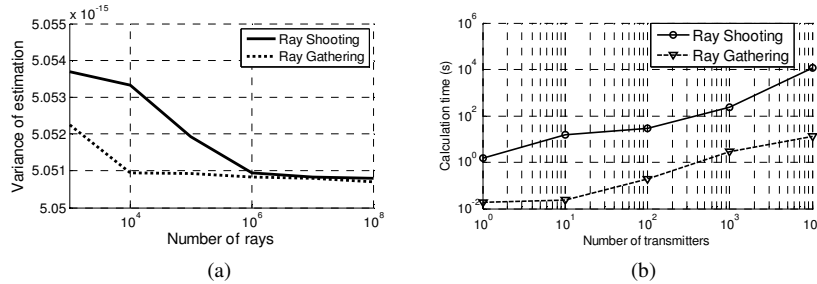


Fig. 7. Study of (a) observed variance according to the sample size (b) computation time according to the number of transmitter

The variance of the Ray Gathering algorithm tends to stabilize more quickly. The convergence is achieved with 10^4 rays, meaning that the distribution of estimated power almost no longer varies around the average

result. On the contrary, the Ray Shooting algorithm reaches convergence at 10^6 rays. This means that our new algorithm is more efficient in terms of convergence speed, according to a single transmitter (i.e., for SISO link).

The second comparison criterion is the calculation time (See Fig. 7 (b)). For this second study, we fix a variance threshold equal to 5.051×10^{-15} , corresponding to the convergence of the two algorithms (See Fig. 7 (a)). For both algorithms, we notice that the simulation times follow almost a linear variation on a logarithmic scale, according to the number of transmitters, with an approximate gap of 10^2 . This means that whatever the number of transmitter is, Ray Gathering algorithm takes 10^2 times less than the Ray Shooting algorithm. And, the more the number of transmitter increases, the more we gain in computation time. This reduction of the computation time is also very useful whether we want to calculate the Outage Probability P_{out} of OWCL [2] according to a threshold criteria, expressed either by SNR (Signal Noise Ratio) or BER (Binary Error Ratio); hence, to obtain a good estimation of P_{out} we need to calculate the total received power for thousands of transmitter positions distributed in the volume of the simulation scene [2], [3]. This means that the four days of simulations can be reduced in a single hour.

5. Conclusion

In this paper, we present two algorithms to efficiently simulate the OWCL in real environments. A mathematical formalization of the Ray Shooting algorithm was realized by applying the potential equation, which manages the distribution of energy in any simulation scene. This allows us to add classical variance reduction techniques reducing the computation times with respect to previous solutions. A new simulation model is also mathematically formalized, by reversing the process of optical wave propagation and launching rays from the receiver. It is more particularly implemented for sensor networks with several transmitters in the context of WBAN. These new two algorithms are validated with respect to reference results [10]. With WBAN, we are mainly interested in the speed of convergence and computation time for our simulators. We show that, according to both these comparison criteria, our second algorithm has significantly higher performance than the others ones. As future works, we plan to develop a new dedicated MIMO configuration algorithm by merging the Ray Shooting and Ray Gathering algorithms.

References

- [1] J.M.Kahn and J.R.Barry, "Wireless infrared communications", Proc IEEE, vol. 85, no. 2, pp. 265-298, Feb. 1997.
- [2] S. S. Torkestani, S. Sahuguede, A. Julien-Vergonjanne, L. Aveneau, P. Combeau, O. Gueu, "Outage Probability Performance in Mobile Indoor Optical Wireless Communication Environment", 1st International Workshop on Optical Wireless Communication, Pisa, Italy, october 2012.
- [3] L. Chevalier, S. Sahuguede, A. Julien-Vergonjanne, P. Combeau, L. Aveneau "Investigation of wireless optical technology for communication between on-body nodes", 2nd International Workshop on Optical Wireless Communication, Northumbria University – Newcastle Upon Tyne, United Kingdom, 79-83, october 2013.
- [4] FJ López-Hernandez, R Pérez-Jiménez, A Santamaría, "Ray-tracing algorithms for fast calculation of the channel impulse response on diffuse IR-Wireless indoor channels", Opt. Eng. 39(10), 2775-2780. 2000.
- [5] Ph. Dutré, "A mathematical framework for global illumination algorithms", In Winter School of Computer Graphics and CAD Systems, pages 33–49, Pilzen, Czech Republic, January 1994.
- [6] E. Lafortune "Mathematical Models and Monte Carlo Algorithms for Physically Based Rendering", Department of Computer Science, Faculty of Engineering, Katholieke Universiteit Leuven. 20-23. 74-79. 1996.
- [7] O Gonzales, S Rodriguez, R Pérez-Jiménez, B Mendoza, A Alaya, "Error analysis of the Simulated Impulse Response on Indoor Wireless Optical Channels Using a Monte Carlo-Based Ray-Tracing Algorithm", IEEE Transactions on Communications 53(1), 124-130. 2005.
- [8] Eric Veach. "Robust Monte Carlo Methods for Light Transport Simulation", PhD. thesis, Stanford University, 1997.
- [9] S. N. Pattanaik and S. P. Mudur. "The potential equation and importance in illumination computations", *Computer Graphics Forum*, 12(2):131–136, 1993.
- [10] John R. Barry, Joseph M. Kahn, William J. Krause, Edward A. Lee, and David G. Messerschmitt. "Simulation of Multipath Impulse Response for Indoor Wireless Optical Channels", IEEE journal on selected areas in communications, vol. 11, no. 3. 1993.